

BELTRAMI FIELDS WITH A NONCONSTANT PROPORTIONALITY FACTOR ARE RARE

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ABSTRACT. We consider the existence of Beltrami fields with a nonconstant proportionality factor f in an open subset U of \mathbb{R}^3 . By reformulating this problem as a constrained evolution equation on a surface, we find an explicit differential equation that f must satisfy whenever there is a nontrivial Beltrami field with this factor. This ensures that there are no nontrivial solutions for an open and dense set of factors f in the C^k topology. In particular, there are no nontrivial Beltrami fields whenever f has a regular level set diffeomorphic to the sphere. This provides an explanation of the helical flow paradox of Morgulis, Yudovich and Zaslavsky (*Comm. Pure Appl. Math.* 48 (1995) 571–582).

1. INTRODUCTION

A *Beltrami field* is a vector field u in \mathbb{R}^3 such that

$$(1.1) \quad \operatorname{curl} u = fu, \quad \operatorname{div} u = 0,$$

where f is a smooth function. The condition that u be divergence-free is redundant when the proportionality factor f is a nonzero constant (i.e., in the case of *strong* Beltrami fields), while otherwise it is tantamount to demanding that the function f be a first integral of u , that is,

$$(1.2) \quad u \cdot \nabla f = 0.$$

Beltrami fields have been studied since the XIX century because of their connection with the Euler equation and with magnetohydrodynamics, where they are known as force-free fields. Indeed, it is well known that a Beltrami field is also a solution of the steady Euler equation in \mathbb{R}^3 ,

$$(u \cdot \nabla)u = -\nabla P, \quad \operatorname{div} u = 0$$

with $P = -\frac{1}{2}|u|^2$, and actually the analysis of concrete examples of Beltrami fields with constant proportionality factor such as the ABC flows [2] has yielded considerable insight e.g. into the phenomenon of Lagrangian turbulence [9].

Beyond the study of explicit examples, Beltrami fields with constant proportionality factor have found application as powerful tools to analyze the structure of solutions to the Euler equation. For instance, de Lellis and Székelyhidi have utilized strong Beltrami fields to construct Hölder continuous weak solutions to the Euler equation in the 3-torus that dissipate energy [7, 8], while in Refs. [10, 11] we constructed strong Beltrami fields in \mathbb{R}^3 having vortex lines and vortex tubes (that is, integral curves and invariant tori) of arbitrary topology. Expansions of more general solutions to the Euler equation in terms of strong Beltrami fields were also considered in [6].

On the contrary, Beltrami fields with nonconstant proportionality factor have not found as many applications, and indeed to the best of our knowledge there are just a handful of explicit examples, all of which have Euclidean symmetries. In fact, the analysis of Beltrami fields with nonconstant factor has proved to be extremely hard, as one can infer from the striking lack of results in this classical subject. An interesting contribution in this direction is the construction of low-regularity Beltrami fields with Hölder-continuous nonconstant factors in [4, 12].

More precisely, the key question, sometimes called the helical flow problem, is to ascertain for which functions f there is a nontrivial vector field satisfying the Eq. (1.1). In this regard, an influential observation due to Morgulis, Yudovich and Zaslavsky [13] is that one would naively expect “most” Beltrami fields to admit a first integral, since this happens whenever the function f is nonconstant as a consequence of Eq. (1.2). This is referred to as the helical flow paradox, and physically means that the fluid flow defined by a Beltrami field would generically be laminar, in contrast with the physical intuition that the fluid should typically present a turbulent behavior [14].

However, since the first integral condition (1.2) is very restrictive, it stands to reason that the Eq. (1.1) should not admit any nontrivial solutions for most functions f . Our objective in this paper is to make precise this idea.

Specifically, our main result asserts that, for a generic function f , the only vector field u satisfying Eq. (1.1) is the trivial one, $u \equiv 0$. This provides an explanation of the helical flow paradox, as it shows that the hypothetical laminar flow associated to a nonconstant proportionality factor does not exist generically. Here “generic” means that f must satisfy certain complicated PDE, so that the set of functions f admitting a nontrivial local Beltrami field are of codimension at least 1 in the C^k topology. This PDE has a complicated expression but can be computed explicitly, so it yields an effective necessary condition for f to admit nontrivial solutions as will be illustrated in Propositions 3.2 and 3.4 below.

Theorem 1.1. *Let $U \subseteq \mathbb{R}^3$ be a domain and assume that the function f is non-constant and of class $C^{6,\alpha}$. Suppose that the vector field u satisfies the Eq. (1.1) in U . Then there is a nonlinear partial differential operator $P \neq 0$, which can be computed explicitly and involves derivatives of order at most 6, such that $u \equiv 0$ unless $P[f]$ is identically zero in U . In particular, $u \equiv 0$ for all f in an open and dense subset of $C^k(U)$ with any $k \geq 7$.*

An easy consequence of the proof of the main result is that if f has a regular level set diffeomorphic to the sphere, then the Eq. (1.1) does not have any nontrivial solutions. In particular, there are no Beltrami fields whenever f has local extrema or is a radial function. This is somehow related to the classical theorem of Cowling ensuring that there are no poloidal Beltrami fields with nonconstant factor and axial symmetry [3].

Theorem 1.2. *Suppose that the function f is of class $C^{2,\alpha}$ in a domain $U \subseteq \mathbb{R}^3$. If a regular level set $f^{-1}(c)$ has a connected component in U diffeomorphic to \mathbb{S}^2 , then any solution to the Eq. (1.1) in U is identically zero.*

Before passing to discuss the proof of these results, a few comments are in order. Firstly, notice that the reason for which we have not made any regularity

assumptions on u is that it automatically satisfies the elliptic equation

$$\Delta u + \nabla f \times u + f^2 u = 0,$$

which ensures that u is of class $C^{k+1,\alpha}$ if f is $C^{k,\alpha}$. Furthermore, this shows that u satisfies the unique continuation property, so u is identically zero in its domain if it vanishes in any open subset. Secondly, an interesting consequence of the proof of these results is that the theorems remain valid if we assume instead that u is a strong Beltrami field, satisfying

$$\operatorname{curl} u = \lambda u$$

for some nonzero constant λ , and f is a first integral in U . Therefore, the first integrals of a strong Beltrami field are also severely restricted. Thirdly, all the results and proofs remain valid for the Beltrami equation in an arbitrary Riemannian 3-manifold, but we have restricted ourselves to Euclidean space to simplify the exposition.

The proof of these theorems, given in Section 3, is based on formulating the Beltrami equation (1.1) as a constrained evolution problem. Although the underlying mathematics are relatively unsophisticated, we regard this reformulation as the main contribution of the paper. Indeed, one can show that the Eq. (1.1) is locally equivalent, in a sense to be made precise later on, to the assertion that there is a time-dependent 1-form $\beta(t)$ on a surface Σ that satisfies the evolution equation

$$(1.3) \quad \partial_t \beta = T(t) \beta$$

together with the constraint

$$(1.4) \quad d\beta = 0.$$

Here $T(t)$ is a time-dependent tensor field that depends on f and the exterior differential d is computed with respect to the coordinates on the surface Σ , which, in turn, is a regular level set of f . Full details are given in Section 2.

This formulation lays bare the reason for which the Beltrami equation does not generally admit nonzero solutions: the evolution (1.3) is not generally compatible with the constraint (1.4), and the resulting compatibility conditions translate into equations that f and its derivatives must satisfy. In Theorems 1.1 and 1.2 we have presented the first two of these compatibility conditions, but in fact there is a whole hierarchy of explicitly computable obstructions (with increasingly cumbersome expressions). Furthermore, it provides an appealing explanation, without even resorting to the statement of the previous theorems, of the reason for which the attempts at constructing solutions to (1.1) using variational techniques have failed: while the regularity of the equation is indeed determined by an elliptic system, its existence is in fact controlled by a constrained evolution problem for which the existence theory is ill posed.

To conclude, let us emphasize that the key to the obstructions for the existence of nontrivial solutions to the Eq. (1.1) is indeed the requirement that u be divergence-free. In fact, in Section 4 we will show that if this condition is omitted, there are always solutions in the whole space \mathbb{R}^3 provided that the function f is positive. This case corresponds to a compressible fluid flow, with f playing the role of the density of the fluid.

2. THE BELTRAMI EQUATION AS A CONSTRAINED EVOLUTION

Our goal in this section is to reformulate the Beltrami equation (1.1) as a constrained evolution problem for a 1-form on a surface. The key equations that we derive here are (2.9) and (2.12), which were already discussed in the Introduction.

Let us take a point p of the domain U such that the gradient of f does not vanish in a small neighborhood of p . Without loss of generality we can assume that this neighborhood is indeed U ,

$$f(p) = 1$$

and

$$\Sigma := f^{-1}(1) \cap U$$

is a connected surface. By rotating the coordinate axes if necessary, Σ can be parametrized as a graph, namely

$$\Sigma = \{(\xi, h(\xi))\}$$

with the coordinates $\xi = (\xi_1, \xi_2)$ taking values in a disc. Here h is a function of the same regularity as f and is defined via the implicit function theorem and the relation

$$f(\xi, h(\xi)) = 1.$$

Moreover, it is standard that we can assume that the point p lies at the origin of the coordinate system and that the gradient of f is parallel to the third coordinate at that point, which means

$$h(0) = 0 \quad \text{and} \quad \frac{\partial h}{\partial \xi_i}(0) = 0.$$

Let us consider the vector field

$$X := \frac{\nabla f}{|\nabla f|^2}$$

and denote by ϕ_t its local flow. We can parametrize U by coordinates (t, ξ) defined via

$$(2.1) \quad x = \phi_t((\xi, h(\xi))).$$

It is clear that $f = 1$ when $t = 0$ by the definition of the function h and that

$$X \cdot \nabla f = 1,$$

which implies that

$$(2.2) \quad f(\phi_s x) = f(x) + s$$

as long as the action of the local flow on x is defined. In particular, we deduce that in the new coordinates the function f reads as

$$(2.3) \quad f = 1 + t.$$

It is important to notice that, in these coordinates, the Euclidean metric is of the form

$$(2.4) \quad ds^2 = \chi(t, \xi)^2 dt^2 + g_{ij}(t, \xi) d\xi_i d\xi_j,$$

where the function χ stands for the function $1/|\nabla f|$ written in the new coordinates and

$$g_{ij} := \partial_i x \cdot \partial_j x$$

is the induced metric of the surface of constant t . Here x is given in terms of (t, ξ) by (2.1) and ∂_i henceforth stands for the derivative with respect to ξ_i . Since $|X| = 1/|\nabla f|$, the only nontrivial assertion here is that the crossed terms

$$\partial_i x \cdot \partial_t x$$

are zero. The easiest way to see this is to prove that the inverse of the metric tensor, which we claim to be of the form

$$\begin{pmatrix} \chi^2 & 0 & 0 \\ 0 & g_{11} & g_{12} \\ 0 & g_{21} & g_{22} \end{pmatrix},$$

is indeed read as

$$\begin{pmatrix} \chi^{-2} & 0 & 0 \\ 0 & g^{11} & g^{12} \\ 0 & g^{21} & g^{22} \end{pmatrix}.$$

This is immediate, for it is well known that the (t, i) component of the latter matrix is precisely

$$\begin{aligned} \nabla f \cdot \nabla \xi_i &= |\nabla f|^2 X \cdot \nabla \xi_i \\ &= |\nabla f|^2 \frac{d}{ds}(\xi_i \circ \phi_s) \\ &= 0. \end{aligned}$$

Here we are considering the variables ξ_i as functions of x and to pass to the last line we have used that, as a consequence of (2.1) and (2.2),

$$\xi_i(\phi_s x) = (\phi_{1-f(\phi_s x)} \circ \phi_s x)_i = (\phi_{1-f(x)} x)_i = \xi_i(x)$$

for all s , with the subscript i denoting the i^{th} component of the point.

Given a solution u to Eq. (1.1) in U , let us denote by β its dual 1-form, computed using the Euclidean metric. The first integral condition (1.2), together with the block structure of the metric in these coordinates shown in Eq. (2.4), then imply that β must be of the form

$$(2.5) \quad \beta = \beta_i(t, \xi) d\xi_i.$$

Denoting by $|g|$ the determinant of the matrix (g_{ij}) , a straightforward computation then shows that the differential and Hodge star of β are as follows:

$$(2.6) \quad d_{\mathbb{R}^3} \beta = (\partial_1 \beta_2 - \partial_2 \beta_1) d\xi_1 \wedge d\xi_2 + \partial_t \beta_1 dt \wedge d\xi_1 + \partial_t \beta_2 dt \wedge d\xi_2,$$

$$(2.7) \quad *_{\mathbb{R}^3} \beta = \chi |g|^{1/2} (g^{2i} \beta_i dt \wedge d\xi_1 - g^{1i} \beta_i dt \wedge d\xi_2).$$

Here we are using the cumbersome notation $d_{\mathbb{R}^3} \beta$ and $*_{\mathbb{R}^3}$ to stress that these operations are computed with respect to all three variables (t, ξ) and thus avoid confusion with the two-dimensional exterior derivative and Hodge operator that we will introduce shortly.

When expressed in terms of the dual 1-form, the Beltrami equation (1.1) takes the form

$$d_{\mathbb{R}^3} \beta = f *_{\mathbb{R}^3} \beta.$$

Reading off the coefficients from (2.6)-(2.7) and using the equation (2.3), the Beltrami equation in the coordinates (t, ξ) amounts to the following system:

$$(2.8a) \quad \partial_t \beta_1 = (1+t)\chi|g|^{1/2} g^{2i} \beta_i,$$

$$(2.8b) \quad \partial_t \beta_2 = -(1+t)\chi|g|^{1/2} g^{1i} \beta_i,$$

$$(2.8c) \quad 0 = \partial_1 \beta_2 - \partial_2 \beta_1.$$

To analyze this system, we begin by making use of Eq. (2.5) to consider β as a time-dependent 1-form on the surface Σ , which maps each “time” t to a 1-form in two dimensions $\beta(t)$. Eqs. (2.8a)-(2.8b) show that the evolution in time of this 1-form is defined by a time-dependent tensor field $T(t)$ on Σ as

$$(2.9) \quad \partial_t \beta = T(t) \beta.$$

In fact, $T(t)$ can be written in terms of the Hodge operator $*_t$ associated with the time-dependent metric

$$g_{ij}(t, \xi) d\xi_i d\xi_j$$

on Σ as

$$(2.10) \quad T(t)\beta = -(1+t)\chi(t, \xi) *_t \beta$$

and its components are

$$(2.11) \quad (T_i^j) = (1+t)\chi|g|^{1/2} \begin{pmatrix} g^{12} & g^{22} \\ -g^{11} & -g^{12} \end{pmatrix}$$

On the contrary, Eq. (2.8c) does not describe an evolution, but impose the stationary constraint that $\beta(t)$ must be closed (as a 1-form on Σ) for all times. Denoting by d the exterior differential on the surface, this reads as

$$(2.12) \quad d\beta = 0.$$

3. PROOF OF THE THEOREMS

To derive a useful necessary condition for β to be a solution of the system (2.9)-(2.12), which is equivalent to the Beltrami equation (1.1), let us begin by defining the family of time-dependent tensor fields $T_n(t)$ recursively as

$$\begin{aligned} T_1 &:= T, \\ T_{n+1} &:= \partial_t T_n + T_n T. \end{aligned}$$

It is not hard to see that $T_n(t)$ depends on n derivatives of f in a non-local manner, the non-locality being due to the definition of the coordinate system, and that $T_n(0)$ is a (local, nonlinear) function of the first n derivatives of f . These tensor fields can be used to describe the constraints of the system due to the following

Proposition 3.1. *If the function f is of class $C^{k,\alpha}$ and $n \leq k-1$, the time-dependent 1-form β must satisfy the constraint*

$$d(T_n \beta) = 0$$

at all times.

Proof. As the constraint equation (2.12) holds for all times, it trivially implies that the time derivatives of the 1-form β must satisfy the constraint

$$d(\partial_t^n \beta) = 0,$$

for any n . An easy induction argument using the evolution equation (2.9) shows that

$$\partial_t^n \beta = T_n \beta,$$

so the proposition follows. \square

By exploiting the previous constraints with $n = 0, 1$, we are now ready to prove that there are no nontrivial Beltrami fields whenever f has a level set diffeomorphic to a sphere:

Proof of Theorem 1.2. Since we are assuming that the surface $\Sigma = f^{-1}(1)$ is a sphere, Eq. (2.3) and the fact that the gradient of f does not vanish on Σ imply that $\Sigma_{t_0} := f^{-1}(1 + t_0)$ is also diffeomorphic to \mathbb{S}^2 for small enough t_0 . Hence the constraint equation (2.12) implies that there is a scalar function $\psi(t, \xi)$ such that

$$\beta = d\psi.$$

By the construction of β and the regularity of f , the 1-form β is of class $C^{1,\alpha}$, so $\psi(t, \xi)$ is a $C^{2,\alpha}$ function of ξ .

Taking into account the form of the tensor field T (cf. Eq. (2.10)), we then find that

$$\begin{aligned} d(T\beta) &= -(1+t)d(\chi *_t d\psi) \\ &= (1+t)\chi \left[-d*_t d\psi - d(\log \chi) \wedge *_t d\psi \right] \\ &= -(1+t)\chi \left[\Delta_t \psi + \langle \nabla_t(\log \chi), \nabla_t \psi \rangle_t \right] \text{Vol}_t, \end{aligned}$$

where the subscripts denote that the Laplacian, gradient and volume form are computed on the sphere Σ_t using the induced metric, which has components $g_{ij}(t, \xi)$. Since χ is nonzero for small enough t , Proposition 3.1 then ensures that the equation

$$\Delta_t \psi + \langle \nabla_t(\log \chi), \nabla_t \psi \rangle_t = 0$$

holds. As this equation satisfies the maximum principle in the closed surface Σ_t , it follows that $\psi(t, \xi)$ is a constant that depends only on t . Thus $\beta \equiv 0$ in a neighborhood of Σ , and therefore everywhere by unique continuation. \square

The proof of Theorem 1.1 also makes crucial use of Proposition 3.1 to show that the function f must satisfy some differential constraint:

Proof of Theorem 1.1. Let us begin by recording the following formula for $d(T_n \beta)$ in local coordinates:

$$\begin{aligned} (3.1) \quad d(T_n \beta) &= \left[(\partial_1(T_n)_2^i - \partial_2(T_n)_1^i) \beta_i + ((T_n)_2^2 - (T_n)_1^1) \partial_2 \beta_1 \right. \\ &\quad \left. + (T_n)_2^1 \partial_1 \beta_1 - (T_n)_1^2 \partial_2 \beta_2 \right] d\xi_1 \wedge d\xi_2. \end{aligned}$$

Here we have used that $\partial_1 \beta_2 = \partial_2 \beta_1$ by the constraint equation (2.8c).

By Eq. (2.11) and the fact that g_{ij} is a metric,

$$T_1^2 = (1+t)\chi|g|^{1/2}g^{22}$$

is strictly positive. Since $d(T\beta) = 0$ by Proposition 3.1, we can therefore isolate $\partial_2\beta_2$ in this equation, finding that

$$(3.2) \quad \partial_2\beta_2 = \frac{1}{T_1^2} \left[(\partial_1 T_2^i - \partial_2 T_1^i) \beta_i + T_2^1 \partial_1 \beta_1 + (T_2^2 - T_1^1) \partial_2 \beta_1 \right].$$

To simplify the notation, let us consider the 4-component vectors

$$\Gamma := (\beta_1, \beta_2, \partial_1 \beta_1, \partial_2 \beta_1)$$

and

$$(3.3) \quad \mathcal{T}_n := \begin{pmatrix} \partial_1 (T_n)_2^1 - \partial_2 (T_n)_1^1 - \frac{(T_n)_1^2}{T_1^2} (\partial_1 T_2^1 - \partial_2 T_1^1) \\ \partial_1 (T_n)_2^2 - \partial_2 (T_n)_1^2 - \frac{(T_n)_1^2}{T_1^2} (\partial_1 T_2^2 - \partial_2 T_1^2) \\ (T_n)_2^1 - \frac{(T_n)_1^2 T_2^1}{T_1^2} \\ (T_n)_2^2 - (T_n)_1^1 - \frac{(T_n)_1^2}{T_1^2} (T_2^2 - T_1^1) \end{pmatrix}$$

Using Eqs. (2.8c) and (3.2) in (3.1), one can then write

$$d(T_n \beta) = (\Gamma \cdot \mathcal{T}_n) d\xi_1 \wedge d\xi_2,$$

where the dot has the obvious meaning. Hence the constraint $d(T_n \beta) = 0$ granted by Proposition 3.1 takes the form

$$(3.4) \quad \mathcal{T}_n \cdot \Gamma = 0,$$

the condition being a priori nontrivial for all $n \geq 2$. In particular, if the Beltrami equation has a nonzero solution, the matrix $(\mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5)$ cannot be of maximal rank, that is,

$$(3.5) \quad \det(\mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5) = 0.$$

Due to the definition of the tensor fields T_n and \mathcal{T}_n , this equation involves derivatives of f of order at most 6.

Eq. (3.5) is almost the differential constraint $P[f] = 0$ whose existence was claimed in the statement of the theorem. The only subtle point is that, as we discussed when we defined the tensor fields T_n , for $t \neq 0$ Eq. (3.5) is not a local function of f because we have used the implicit function theorem and the flow of the vector field X to construct the local coordinate system (t, ξ) . However the differential constraint can be defined on the initial surface Σ as

$$(3.6) \quad P[f]|_\Sigma := \det(\mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5)|_{t=0},$$

and this is indeed a local function of f and its derivatives up to sixth order. Since the initial surface Σ is arbitrary and the dependence on the surface is smooth, by carrying out the same construction with Σ replaced by Σ_{t_0} , for all t_0 in an interval around 0, this defines $P[f]$ in a neighborhood of Σ , thereby completing the proof of the theorem. \square

To show that the nonlinear differential operator P is nontrivial, we will approximately compute its action on a couple of concrete functions f . As a byproduct, we will provide a few simple, explicit examples of functions for which there are no nontrivial solutions to the Beltrami equation (1.1) and illustrate how one can evaluate $P[f]$ in practice. More sophisticated examples can be obviously obtained using the same procedure.

Proposition 3.2. *Suppose that the vector field u satisfies the Beltrami equation (1.1) with*

$$f(x) := 1 + ax_1 + bx_1^3 + x_3$$

in a neighborhood of the origin. Then $u \equiv 0$ if $b \neq 0$.

Proof. Using the same notation as in the proof of Theorem 1.1, we can parametrize $\Sigma := f^{-1}(1)$ as the graph

$$\Sigma = \{(\xi, h(\xi))\},$$

with

$$h(\xi) := -a\xi_1 - b\xi_1^3.$$

After a lengthy but straightforward computation starting from Eq. (2.1), one can compute the remaining objects that appear in the proof of Theorem 1.1 as a power series in the coordinate t . In particular, one finds that the determinant (3.6) takes the form

$$P[f]_\Sigma = \sum_{j=0}^4 c_j \xi_1^j + O(\xi_1^5),$$

where the coefficients c_j depend on a and b as:

$$\begin{aligned} c_0 &:= -\frac{5184a^2b^4(15a^4 + 14a^2 + 36ab - 1)}{(a^2 + 1)^{14}}, \\ c_1 &:= -\frac{20736a^2b^4(8a^3 - 63a^2b + 8a - 9b)}{(a^2 + 1)^{14}}, \\ c_2 &:= \frac{31104ab^5(169a^6 + 97a^4 + 468a^3b - 73a^2 - 36ab - 1)}{(a^2 + 1)^{15}}, \\ c_3 &:= \frac{124416a^2b^5(84a^4 - 771a^3b + 68a^2 - 15ab - 16)}{(a^2 + 1)^{15}}, \\ c_4 &:= 46656b^6 + abG. \end{aligned}$$

Here G is a complicated smooth rational function of a and b that can be computed explicitly.

The point now is that the only solution to the system of algebraic equations

$$c_j = 0 \quad \text{for } 0 \leq j \leq 4$$

is $b = 0$. In order to see this, a simple computation shows that imposing $c_j = 0$ for $0 \leq j \leq 3$ implies that $ab = 0$, while $c_4 = 0$ for $b = 0$ but not for $a = 0$. The proposition then follows from Theorem 1.1. \square

Remark 3.3. For $b = 0$, the function f is affine and the Beltrami equation (1.1) does admit a nontrivial solution, which is in fact defined in the whole space. Specifically, if u_0 is any vector in \mathbb{R}^3 orthogonal to $e := (a, 0, 1)$,

$$u := u_0 \cos \frac{(1 + ax_1 + x_3)^2}{2|e|} + \frac{u_0 \times e}{|e|} \sin \frac{(1 + ax_1 + x_3)^2}{2|e|}$$

is such a solution. Hence the set of obstructions on f that we get from the operator P is optimal for this family of functions.

Proposition 3.4. *Suppose that the vector field u satisfies the Beltrami equation (1.1) with*

$$f(x) := 1 + x_1^2 + ax_2^2 + x_3$$

in a neighborhood of the origin. Then $u \equiv 0$ if $a \neq 1$.

Proof. As before, we parametrize $\Sigma := f^{-1}(1)$ as the graph

$$\Sigma = \{(\xi, h(\xi))\},$$

with

$$h(\xi) := -\xi_1^2 - a\xi_2^2.$$

Arguing as in Proposition 3.2 one finds that the determinant (3.6) is of the form

$$\begin{aligned} P[f]|_{\Sigma} = & 1024(a-1)^2 \left[(33 + 128a + 312a^2 + 224a^3 + 768a^4 - 256a^5)\xi_1^2 \right. \\ & - 16a^2(3 + 11a + 66a^2 - 88a^3 + 8a^4)\xi_1\xi_2 \\ & \left. + a^4(-39 - 24a + 760a^2 + 640a^3 - 128a^4)\xi_2^2 \right] + O(|\xi|^3). \end{aligned}$$

It can be easily checked that the quadratic part of this function vanishes if and only if $a = 1$, so the proposition follows from Theorem 1.1. \square

4. FINAL REMARKS

Let us conclude with a few comments regarding the existence of Beltrami flows, in view of the results we have established in this paper.

4.1. Compressible Euler flows. In Ref. [13], considerable attention is paid to the bearing of compressible Beltrami fields on the helical flow paradox. Using Theorem 1.1 and the results that we proved in [10] we can now show that compressible Beltrami fields have totally different existence properties than the incompressible ones, since whenever the function f does not change sign one has many associated solutions.

More precisely, we have the following theorem. We recall that a compressible Beltrami field is not a solution of the Euler equation unless the barotropic condition is satisfied, and that it is natural to assume that f is positive because it plays the role of the fluid density.

Theorem 4.1. *Let f be a positive real-analytic function in \mathbb{R}^3 . Then there are nontrivial solutions to the equation*

$$(4.1) \quad \operatorname{curl} u = fu$$

defined in the whole space \mathbb{R}^3 .

Proof. We proved in [10, Example 8.2] that if \tilde{g} is an analytic (possibly incomplete) Riemannian metric in \mathbb{R}^3 , there is a vector field v , not identically zero, which satisfies the equation

$$\operatorname{curl}_{\tilde{g}} v = v$$

in \mathbb{R}^3 . Here $\operatorname{curl}_{\tilde{g}}$ denotes the curl operator associated with the metric \tilde{g} .

Let us now choose \tilde{g} as the conformally flat metric

$$\tilde{g} := f^2 g_0,$$

where g_0 denotes the Euclidean metric. If we set $u := f^2 v$, a straightforward computation shows that the Euclidean curl of u is given by

$$\operatorname{curl} u = f u,$$

thus completing the proof of the theorem. \square

Remark 4.2. In particular, a straightforward consequence of [10, Example 8.2] and of the proof of the theorem is the following: if f is a positive analytic function and L is any locally finite link in \mathbb{R}^3 , one can transform it using a smooth diffeomorphism Φ of \mathbb{R}^3 so that $\Phi(L)$ is a set of vortex lines of the vector field u , which satisfies Eq. (4.1) in \mathbb{R}^3 . Furthermore, Φ can be chosen close to the identity in any C^k norm. Hence, there is much freedom in the choice of the nontrivial solution u .

4.2. Strong Beltrami fields. When the function f equals some nonzero constant λ , it is well known that the Beltrami equation (1.1) has an infinite number of solutions. In particular, if c_{lm} is a set of constant vectors in \mathbb{R}^3 for which the sum

$$u := \operatorname{curl}(\operatorname{curl} + \lambda) \sum_{l=0}^{\infty} \sum_{m=-l}^l c_{lm} j_l(\lambda r) Y_{lm}(\theta, \varphi)$$

converges in a suitable sense, u is a Beltrami field with constant λ . Here we are using spherical coordinates, j_l is the spherical Bessel function and Y_{lm} are the spherical harmonics.

However, the results we have proved in this paper also have an implication about strong Beltrami fields. In fact, it can be readily checked that the proofs of the main results remain valid under the assumption that u is a strong Beltrami field and f is a first integral of u . Hence we get for free the following

Theorem 4.3. *Assume that u is a strong Beltrami field in a domain $U \subseteq \mathbb{R}^3$. Then it cannot have a first integral of class $C^{2,\alpha}(U)$ with a regular level set diffeomorphic to \mathbb{S}^2 . Furthermore, a (nonconstant) function $f \in C^{6,\alpha}(U)$ cannot be a first integral of u unless it satisfies the equation $P[f] = 0$, where P is a nonlinear differential operator of sixth order that does not depend on the particular Beltrami field u and which can be computed explicitly.*

Notice that the assertion that a first integral of a Beltrami field cannot have a level set diffeomorphic to the sphere is reminiscent of (and somehow complementary to) Arnold's structure theorem [1] for steady solutions of the Euler equation with nonconstant Bernoulli function (that is, for solutions where u and $\operatorname{curl} u$ are not collinear). In this case, the compact level sets of the Bernoulli function must be tori.

4.3. Further differential constraints. We saw in the proof of Theorem 1.1 that the differential operator P that yields the constraints for the function f is given in terms of the 4-component vectors \mathcal{T}_n defined in (3.3) via

$$P[f]|_{\Sigma} := \det(\mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5)|_{t=0}.$$

An important observation is that the proof of Theorem 1.1 actually gives more information than the statement of the theorem. Actually, it is a straightforward

consequence of Eq. (3.4) that f must also satisfy the differential equation $P_{ijkl}[f] = 0$, where we set

$$P_{ijkl}[f]|_{\Sigma} := \det(\mathcal{T}_i, \mathcal{T}_j, \mathcal{T}_k, \mathcal{T}_l)|_{t=0}$$

for any integers

$$l > k > j > i \geq 2,$$

provided that f is smooth enough (e.g., of class $C^{l+1,\alpha}$). Therefore, one would expect to have a hierarchy of differential constraints on a smooth f to admit non-trivial solutions. Notice that proving the independence of the resulting system of constraints should be a delicate problem due to the complexity of the expressions for P_{ijkl} .

4.4. Steady Euler flows in two dimensions. Let us consider a vector field $v = \nabla^{\perp}\psi$, with ψ a scalar function. It is standard that v is a steady solution of the incompressible Euler equation in two dimensions if the stream function ψ satisfies the equation

$$\Delta\psi = F(\psi)$$

for some function F . It is well known that this equation always admits a nontrivial solution ψ for smooth enough F in any ball that is sufficiently small.

Therefore, it stems that there are no local obstructions to the existence of steady Euler flows in two dimensions for any smooth function F . This is in sharp contrast with the Beltrami solutions of the steady Euler equation in \mathbb{R}^3 (cf. Theorem 1.1). It is worth mentioning that an important recent contribution to the study of the geometry of the space of steady solutions in two-dimensional domains is [5], where Arnold's approach to Euler flows using volume-preserving diffeomorphisms is revisited.

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